Summer Calculus Review

Many students find first-year calculus difficult. The main reason is that in mathematics, as in music or athletics, the development of knowledge and skill is cumulative: what you learn next depends heavily on retention of what you learned before. Facility with basic algebra is very important, as is ability to combine techniques from several areas. It is easy to fall behind and difficult to catch up, especially if your skills have diminished over the summer.

In order to ease the transition from pre-calculus to AP Calculus, we suggest you keep up your mathematical knowledge and skills during the summer. We have compiled a selection of exercises, called '**Are You Ready for Calculus?**'. These cover the parts of the high school curriculum most essential as background for calculus. Many of the problems are challenging. It will take some time to work through the complete set, but do not become discouraged if you have difficulty with some of them. You should consider working with friends or in small groups.

You must learn how to solve the problems in the exercises to have the skills necessary for AP Calculus. Do not start the exercises on August 25th. You need to review these during the summer in a steady process and not in a rush session. Most students who have problems in AP Calculus do not have them due to Calculus theory but instead due to the material, especially Algebra, leading up to Calculus.

ARE YOU READY FOR CALCULUS?

1. Simplify each of the following expressions:

   (a) $\dfrac{x^3 - 9x}{x^2 - 7x + 12}$

   (b) $\dfrac{x^2 - 2x - 8}{x^3 + x^2 - 2x}$

   (c) $\dfrac{\frac{1}{x} - \frac{1}{5}}{\frac{1}{x^2} - \frac{1}{25}}$

   (d) $\dfrac{9 - x^{-2}}{3 + x^{-1}}$

2. Rationalize the denominator in each expression:

   (a) $\dfrac{2}{\sqrt{3} + \sqrt{2}}$

   (b) $\dfrac{4}{1 - \sqrt{5}}$

   (c) $\dfrac{1}{1 + \sqrt{3} - \sqrt{5}}$

3. Write each of the following expression in the form ca^pb^q where, c, p and q are numbers:

   (a) $\dfrac{(2a^2)^3}{b}$

   (b) $\sqrt{9ab^3}$

   (c) $\dfrac{a(2/b)}{3/a}$

   (d) $\dfrac{ab - a}{b^2 - b}$

   (e) $\dfrac{a^{-1}}{(b^{-1})\sqrt{a}}$

   (f) $\left(\dfrac{a^{2/3}}{b^{1/2}}\right)^2 \left(\dfrac{b^{3/2}}{a^{1/2}}\right)$

4. In each equation, solve for x (without using a calculator):

   (a) $5^{(x+1)} = 25$

   (b) $\dfrac{1}{3} = 3^{2x+2}$

   (c) $\log_2 x = 3$

   (d) $\log_3 x^2 = 2\log_3 4 - 4\log_3 5$

5. Simplify each expression:

(a) $\log_2 5 + \log_2(x^2-1) - \log_2(x-1)$

(b) $2\log_4 9 - \log_2 3$

(c) $3^{2\log_3 5}$

6. Simplify each expression:

(a) $\log_{10}(10^{1/2})$

(b) $\log_{10}\left(\frac{1}{10^x}\right)$

(c) $2\log_{10}\sqrt{x} + 3\log_{10} x^{1/3}$

7. Solve the following equations for the indicated variables:

(a) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, for a

(b) $V = 2(ab + bc + ca)$, for a

(c) $A = 2\pi r^2 + 2\pi rh$, for positive r

(d) $A = P + nrP$, for P

(e) $2x - 2yd = y + xd$, for d

(f) $\frac{2x}{4x} + \frac{1-x}{2} = 0$, for x

8. For each of the following equations, complete the square and reduce to one of the standard forms: $y - b = A(x-a)^2$ or $x - a = A(y-b)^2$.

(a) $y = x^2 + 4x + 3$

(b) $3x^2 + 3x + 2y = 0$

(c) $9y^2 - 6y - 9 - x = 0$

9. Factor each expression completely:

(a) $x^6 - 16x^4$

(b) $4x^3 - 8x^2 - 25x + 50$

(c) $8x^3 + 27$

(d) $x^4 - 1$

10. Find *all* real solutions to each equation:

(a) $x^6 - 16x^4 = 0$

(b) $4x^3 - 8x^2 - 25x + 50 = 0$

(c) $8x^3 + 27 = 0$

11. Solve for x in each equation:

(a) $3\sin^2 x = \cos^2 x; \quad 0 \le x < 2\pi$

(b) $\cos^2 x - \sin^2 x = \sin x; \quad -\pi < x \le \pi$

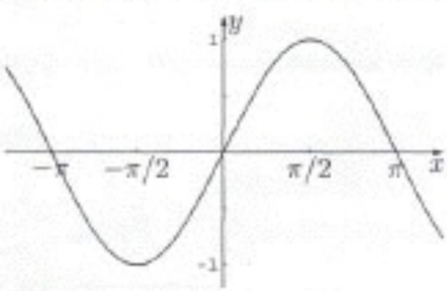
(c) $\tan x + \sec x = 2\cos x; \quad -\infty < x < \infty$

12. Without using a calculator, evaluate the following:

(a) $\cos 210^\circ$ (b) $\sin \frac{5\pi}{4}$ (c) $\tan^{-1}(-1)$ (d) $\sin^{-1}(-1)$

(e) $\cos \frac{9\pi}{4}$ (f) $\sin^{-1} \frac{\sqrt{3}}{2}$ (g) $\tan \frac{7\pi}{6}$ (h) $\cos^{-1}(-1)$

13. Given the graph of $\sin x$, sketch a graph of each of the following:



(a) $y = \sin\left(x - \frac{\pi}{4}\right)$

(b) $y = \sin\left(\frac{x}{2}\right)$

(c) $y = 2\sin x$

(d) $y = \cos x$

(e) $y = \frac{1}{\sin x}$

14. Solve each equation:

(a) $4x^2 + 12x + 3 = 0$

(b) $2x + 1 = \frac{5}{x+2}$

(c) $\frac{x+1}{x} - \frac{x}{x+1} = 0$

15. Find the remainder in each of the following division problems:

(a) $x^5 - 4x^4 + x^3 - 7x + 1$ by $x + 2$

(b) $x^5 - x^4 + x^3 + 2x^2 - x + 4$ by $x^3 + 1$

16. (a) The equation $12x^3 - 23x^2 - 3x + 2 = 0$ has a solution $x = 2$. Find all other solutions.

    (b) Solve for x in the equation $12x^3 + 8x^2 - x - 1 = 0$. (All solutions are rational and between ± 1.)

17. Solve each of the following inequalities:

    (a) $x^2 + 2x - 3 \leq 0$

    (b) $\dfrac{2x-1}{3x-2} \leq 1$

    (c) $x^2 + x + 1 > 0$

18. Solve for x in each equation:

    (a) $|-x + 4| \leq 1$

    (b) $|5x - 2| = 8$

    (c) $|2x + 1| = x + 3$

19. Determine an equation of the following lines:

    (a) The line through (-1,3) and (2,-4).

    (b) The line through (-1,2) and perpendicular to the line $2x - 3y + 5 = 0$.

    (c) The line through (2,3) and the midpoint of the line segment from (-1,4) to (3,2).

20. (a) Find the point of intersection of the lines: $3x - y - 7 = 0$ and $x + 5y + 3 = 0$.

    (b) Shade the region in the xy-plane that is described by the inequalities: $3x - y - 7 < 0$ and $x + 5y + 3 \geq 0$.

21. Find the equations of the following circles:

    (a) The circle with center at $(1, 2)$ that passes through the point $(-2, -1)$.

    (b) The circle that passes through the origin and has intercepts equal to 1 and 2 on the x- and y-axes, respectively.

22. For the circle $x^2 + y^2 + 6x - 4y + 3 = 0$, find:

    (a) The center and the radius.

    (b) The equation of the tangent line at the point $(-2, 5)$.

23. A circle is tangent to the y-axis at $y = 3$ and has one x-intercept at $x = 1$.

    (a) Determine the other x-intercept.

    (b) Find the equation of the circle.

24. A curve is traced by a point $P(x, y)$ which moves such that its distance from the point $A(-1, 1)$ is three times its distance from the point $B(2, -1)$. Determine the equation of the curve.

25. (a) Find the domain of the function $f(x) = \dfrac{3x+1}{\sqrt{x^2+x-2}}$.

(b) Find the domain and range of the functions: i) $f(x) = 7$ ii) $g(x) = \dfrac{5x-3}{2x+1}$.

26. Let $f(x) = \dfrac{|x|}{x}$. Show that $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$. Find the domain and range of $f(x)$.

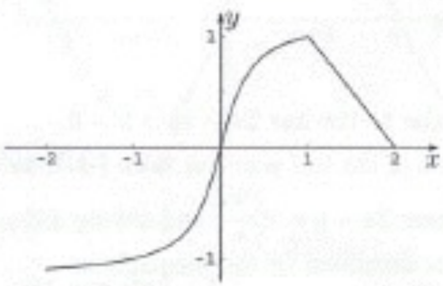
27. Simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$, where

(a) $f(x) = 2x + 3$

(b) $f(x) = \dfrac{1}{x+1}$

(c) $f(x) = x^2$.

28. The graph of the function $y = f(x)$ is given as follows:



Carefully sketch a graph of each of the following:

(a) $y = f(x+1)$

(b) $y = f(-x)$

(c) $y = |f(x)|$

(d) $y = f(|x|)$

29. Carefully sketch a graph of each of the following:

(a) $g(x) = |3x + 2|$

(b) $h(x) = |x(x-1)|$

30. (a) The graph of a quadratic function (a parabola) has x-intercepts -1 and 3 and a range consisting of all numbers less than or equal to 4. Determine an expression for the function.

(b) Sketch the graph of the quadratic function $y = 2x^2 - 4x + 3$.

31. Write each pair of equations as a single equation in x and y:

(a) $\begin{cases} x = t + 1 \\ y = t^2 - t \end{cases}$

(b) $\begin{cases} x = \sqrt[3]{t} - 1 \\ y = t^2 - t \end{cases}$

(c) $\begin{cases} x = \sin t \\ y = \cos t \end{cases}$

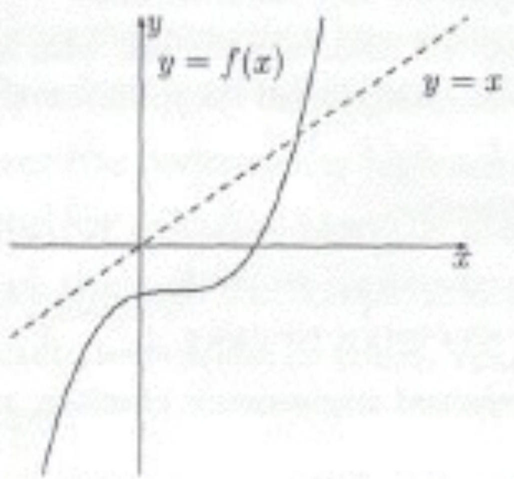
32. Find the inverse of each function:

(a) $f(x) = 2x + 3$

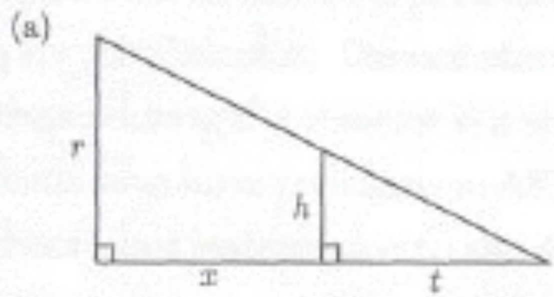
(b) $f(x) = \dfrac{x+2}{5x-1}$

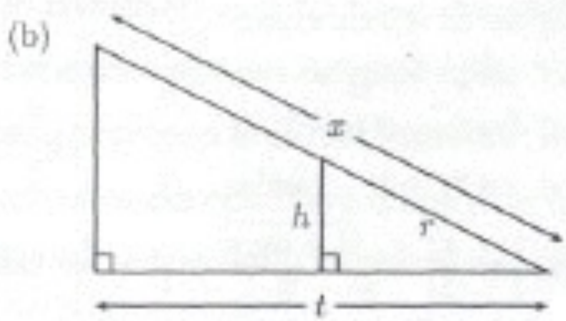
(c) $f(x) = x^2 + 2x - 1,\ x > 0$

33. A function $f(x)$ has the graph given below. Carefully sketch the graph of the inverse function $f^{-1}(x)$.



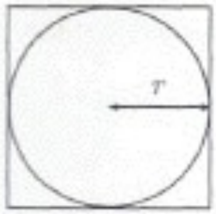
34. Express x in terms of the other variables in the picture.



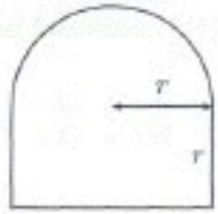


35. Consider the following diagrams:

(A)



(B)



(a) Find the ratio of the area inside the square but outside the circle to the area of the square in the figure (A).

(b) Find the formula for the perimeter of a window of the shape in Figure (B).

(c) A water tank has the shape of a cone (like an ice cream cone without ice cream). The tank is 10m high and has a radius of 3m at the top. If the water is 5m deep (in the middle) what is the surface area of the top of the water?

(d) Two cars start moving from the same point. One travels south at 100 km/hour, the other west at 50 km/hour. How far apart are they two hours later?

(e) A kite is 100m above the ground. If there are 200m of string out, what is the angle between the string and the horizontal. (Assume that the string is perfectly straight).

36. You should know the following trigonometric identities.

(A) $\sin(-x) = -\sin x$

(B) $\cos(-x) = \cos x$

(C) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

(D) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Use these equalities to derive the following *important* trigonometric identities, which you should also know.

(a) $\sin^2 x + \cos^2 x = 1$ (use (C) and $\cos 0 = 1$.)

(b) $\sin 2x = 2 \sin x \cos x$

(c) $\cos 2x = \cos^2 x - \sin^2 x$

(d) $\cos 2x = 2\cos^2 x - 1$

(e) $\cos 2x = 1 - 2\sin^2 x$

(f) $\left|\cos \frac{x}{2}\right| = \sqrt{\frac{1+\cos x}{2}}$

(g) $\left|\sin \frac{x}{2}\right| = \sqrt{\frac{1-\cos x}{2}}$